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Existence Theory for the Boltzmann Equation

ABSTRACT. — The paper presents a short survey of the existence theory for the nonlinear Boltzmann equation with a particular attention for a recent result by the author concerning Maxwell molecules, without any cutoff in the collision kernel, in the one-dimensional case.

1. - INTRODUCTION

The existence theory of the Boltzmann equation is deeply related to proving a rigorous validity result for this equation, starting from the basic laws of mechanics. The importance of the latter issue is evident: we have to settle the fundamental question whether the *irreversible* Boltzmann equation can be rigorously obtained from *reversible* mechanics. The answer to this query is yes; in particular, there is no contradiction between the second law of thermodynamics and the reversibility of molecular dynamics, provided we take the so-called Boltzmann–Grad limit (number of particles $N \to \infty$, particle diameter $\sigma \to 0$, $N\sigma^2$ finite). For short times, this was first proved by O. Lanford in a classical paper [12]. Lanford's paper contains only a sketch of the proof; a more detailed treatment is given in two monographs [7, 14]. The first of these texts presents also an extension [10, 11], due to Illner and Pulvirenti, which gives a global validity result for a gas cloud in all space if the mean free path is large. This is the only global result known so far. In both cases, one can prove propagation of chaos and existence and uniqueness of solutions for the Boltzmann equation.

The well-posedness of the initial value problem for the Boltzmann equation means that we prove that there is a unique nonnegative solution preserving the energy and satisfying the entropy inequality, from a positive initial datum with finite energy and entropy. However, for general initial data, it is difficult, and until now not known, whether such a well-behaved solution can be constructed globally in time. The difficulty

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in doing this is obviously related to the nonlinearity of the collision operator and the apparent lack of conservation laws or a priori estimates preventing the solution from becoming singular in finite time.

A complete validity discussion for the Boltzmann equation will automatically contain existence and uniqueness results. Consequently, by Lanford's theorem, we already have some existence and uniqueness theorems. Unfortunately, a validity proof involves several hard additional steps beyond existence and uniqueness. Therefore, the Boltzmann equation has been validated rigorously only in the few simple situations which we have mentioned above (locally in time, and globally for a rare gas cloud in all space).

Existence (and in some situations uniqueness) of solutions to IVP is known for a much larger variety of cases, and it is our purpose in the next section to survey these results.

2. - A SURVEY OF THE EXISTENCE THEORY

The Boltzmann equation reads as follows:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{x}} = Q(f,f) \tag{2.1}$$

$$Q(f,f)(x,\boldsymbol{v},t) = \int\int B(\boldsymbol{n}\cdot(\boldsymbol{v}-\boldsymbol{v}_*), |\boldsymbol{v}-\boldsymbol{v}_*|)\,(f'f'_* - ff_*)\,\sin\theta\, d\theta d\phi\, d\boldsymbol{v}_*. \tag{2.2}$$

where $f = f(\boldsymbol{x},\boldsymbol{v},t)$ is the probability density of finding a gas molecule at $\boldsymbol{x}$ with velocity $\boldsymbol{v}$ at time t. We denote by $f_*\, f(\boldsymbol{x},\boldsymbol{v}_*,t)$, where $\boldsymbol{v}_*$ is the velocity of a partner in a collision and $f' = f(\boldsymbol{x},\boldsymbol{v}',t)$, $f'_* = f(\boldsymbol{x},\boldsymbol{v}'_*,t)$, where

$$\begin{aligned} \boldsymbol{v}' &= \boldsymbol{v} - \boldsymbol{n}[\boldsymbol{n}\cdot(\boldsymbol{v}-\boldsymbol{v}_*)] \\ \boldsymbol{v}'_* &= \boldsymbol{v}_* + \boldsymbol{n}[\boldsymbol{n}\cdot(\boldsymbol{v}-\boldsymbol{v}_*)]. \end{aligned} \tag{2.3}$$

Here $\boldsymbol{n}$ is the unit vector associated with the angles θ and ϕ. The quadratic operator $Q(.,.)$ is called the collision operator.

The kernel B depends on the molecular model. If the molecules are hard spheres of diameter σ, then

$$B(\boldsymbol{n}\cdot(\boldsymbol{v}-\boldsymbol{v}_*), |\boldsymbol{v}-\boldsymbol{v}_*|) = N\sigma^2|\boldsymbol{v}-\boldsymbol{v}_*|\sin\theta\cos\theta. \tag{2.4}$$

If the molecules are point masses interacting with a force varying as the n-th inverse power of the distance, then

$$B(\boldsymbol{n}\cdot(\boldsymbol{v}-\boldsymbol{v}_*), |\boldsymbol{v}-\boldsymbol{v}_*|) = B(\theta)|\boldsymbol{V}|^{\frac{n-5}{n-1}} \tag{2.5}$$

where $B(\theta)$ is a non-elementary function of θ which for θ close to $\pi/2$ behaves as the power $-(n+1)/(n-1)$ of $|\pi/2-\theta|$. In particular, for $n=5$ one has the Maxwell molecules, for which the dependence on $V = |\boldsymbol{v}-\boldsymbol{v}_*|$ disappears. For a detailed explanation of the structure of the collision term, see Refs. 3, 7, or 8.

Sometimes the artifice of cutting the grazing collisions corresponding to small values of $\left|\theta - \frac{\pi}{2}\right|$ is used (angle cutoff). In this case one has both the advantage of being able to split the collision term and of preserving a relation of the form (2.5) for power-law potentials. This artifice is common in most existence theorems; when it is not stated otherwise, one considers either hard spheres or models with angular cutoff.

When the distribution function of a gas is not depending on the space variables, the equation is considerably simplified. The collision operator is basically Lipschitz continuous in L^1_+ and the equation becomes globally solvable in time. Moreover, uniqueness, asymptotic behavior and a theory of classical solutions have been established. The theory for the spatially homogeneous Boltzmann equation begins in the early thirties and can be considered rather complete; unfortunately the homogeneous case is hardly of interest for applications. The reader interested in this theory should consult Ref. 7.

The Boltzmann equation has a family of equilibrium solutions, the Maxwellians

$$f = A\exp\left(-\beta|\boldsymbol{v} - \boldsymbol{u}|^2\right) \quad (A, \beta, \boldsymbol{u} \quad \text{constants}). \tag{2.6}$$

If the solution is initially sufficiently close to a Maxwellian, it is possible to prove that a solution can be constructed globally in time, and we have uniqueness and asymptotic behavior. The approach is based on the analysis of the linearized Boltzmann operator, which leads us to a differential inequality of the type

$$\frac{d}{dt}y \leq -ky + y^2$$

where $y = y(t)$ is some norm of the deviation of the solution from the Maxwellian and k is a positive number. Therefore, if $y(0)$ is sufficiently small, we can control the solution for all times. As we said, the basic ingredient is good control of the linearized Boltzmann operator. This theory is discussed in Ref. 7.

The case of perturbation of a vacuum is a consequence of the validity result, as mentioned in the previous section, together with the local existence theory. If the initial value is close to a homogeneous distribution, a solution starting from it can be constructed globally in time. Uniqueness and asymptotic behavior can also be proved. The main idea is explained in Ref. 7.

Except the first one, all these results have a perturbation character. The knowledge of particular solutions helps to construct other solutions which are close to the original ones. The general initial value problem is poorly understood, although a significant and somewhat unexpected step was performed in the late eighties. Consider an equation similar to the Boltzmann equation for which we have conservation of mass and energy and the H-Theorem ($\int f \log f d\boldsymbol{x} d\boldsymbol{v}$ cannot grow in time). Denote by $f^\varepsilon(t)$ the solutions. Here, ε is a regularization parameter such that the solutions formally converge to a solution of the Boltzmann equation in the limit $\varepsilon \to 0$. The conservation laws yield the existence of a weak limit denoted by $f(t)$. However, since the collision operator is quadratic in f, it cannot be weakly continuous. Thus it does not follow by general arguments that $f(t)$ solves the Boltzmann equation. Nevertheless, some smoothness gained by the streaming operator gives enough compactness to prove that $f(t)$ actually solves the Boltzmann equation in the mild sense.

The method gives neither uniqueness nor energy conservation, but the entropy is seen to decrease along the solution trajectories.

Let us begin with some notation and a rather standard definition. Let Λf denote the left-hand side of the Boltzmann equation and

$$f^{\#}(\boldsymbol{x},\boldsymbol{v},t) = f(\boldsymbol{x}+\boldsymbol{v}t,\, \boldsymbol{v},t).$$

DEFINITION 2.1: *A measurable function $f = f(\boldsymbol{x},\boldsymbol{v},t)$ on $[0,\infty) \times \Re^3 \times \Re^3$ is a mild solution of the Boltzmann equation for the (measurable) initial value $f_0(\boldsymbol{x},\boldsymbol{v})$ if for almost all $(\boldsymbol{x},\boldsymbol{v})$ $Q_{\pm}(f,f)^{\#}(\boldsymbol{x},\, \boldsymbol{v},\, \cdot)$ are in $L^1_{\text{loc}}[0,\infty)$, and if for each $t \geq 0$*

$$f^{\#}(\boldsymbol{x},\boldsymbol{v},t) = f_0(\boldsymbol{x},\boldsymbol{v}) + \int_0^t Q(f,f)^{\#}(\boldsymbol{x},\, \boldsymbol{v},\, s)\, ds. \tag{2.7}$$

One of the key ideas used by DiPerna and Lions to prove a general existence theorem was to relax the solution concept even further, such that the bounds provided by the energy conservation and the H-theorem could be put to the best use, and then to regain mild solutions via a limit procedure. They called the relaxed solution concept "renormalized solution" and defined it in the following way:

DEFINITION 2.2: *A function $f = f(\boldsymbol{x},\boldsymbol{v},t) \in L^1_+(\Re^+_{\text{loc}} \times \Re^3 \times \Re^3)$ is called a renormalized solution of the Boltzmann equation if*

$$\frac{Q_{\pm}(f,f)}{1+f} \in L^1_{\text{loc}}(\Re_+ \times \Re^3 \times \Re^3 \tag{2.8}$$

and if for every Lipschitz continuous function $\beta : \Re_+ \to \Re$ which satisfies $|\beta'(t)| \leq$ $\leq C/(1+t)$ for all $t \geq 0$ one has

$$\Lambda\beta(f) = \beta'(f)Q(f,f) \tag{2.9}$$

in the sense of distributions.

DiPerna and Lions [8] noticed that renormalization would actually give mild solutions, according to the following

LEMMA 2.1: *Let $f \in (L^1_{\text{loc}} \times \Re^3 \times \Re^3)$. If f satisfies (3.1) and (3.2) with $\beta(t) = \ln(1+t)$, then f is a mild solution of the Boltzmann equation. If f is a mild solution of the Boltzmann equation and if $Q_{\pm}(f,f)/(1+f) \in L^1_{\text{loc}}(\Re_+ \times \Re^3 \times \Re^3)$, then f is a renormalized solution.*

Their main result is:

THEOREM 2.2: (DiPerna and Lions [8]). *Suppose that* $f_0 \in L^1_+(\Re^3 \times \Re^3$ *is such that*

$$\int_{\Re^3}\int_{\Re^3} f_0(1+|\boldsymbol{x}|^2+|\boldsymbol{v}|^2)\, d\boldsymbol{x}\, d\boldsymbol{v} < \infty$$

and

$$\int_{\Re^3}\int_{\Re^3} f_0|\ln f_0|\, d\boldsymbol{x}\, d\boldsymbol{v} < \infty.$$

Then there is a renormalized solution of the Boltzmann equation such that $f \in C(\Re_+, L^1(\Re^3, \Re^3), f\,|_{t=0} = f^0$.

The existence theorem of DiPerna and Lions is rightly considered as a basic result of the mathematical theory of the Boltzmann equation. Unfortunately, it is far from providing a complete theory, since there is no proof of uniqueness; in addition, there is no proof that energy is conserved and conservation of momentum can be proved only globally and not locally. More complete results concerning conservation equations have been recently obtained in the case of solutions depending on just one space coordinate [2, 4].

3. - THE WEAK FORM OF THE COLLISION OPERATOR AND A USEFUL IDENTITY

We introduce the weak form of the collision term, $Q(f,f)$. We shall henceforth use the latter notation for the operator defined by:

(3.1) $$\int_{[0,T]\times[0,1]\times\mathbb{R}^3} Q(f,f)(x,\boldsymbol{v},t)\varphi(x,\boldsymbol{v},t)d\boldsymbol{v}dxdt =$$

$$\frac{1}{2}\int_{[0,T]\times[0,1]\times\mathbb{R}^3\times\mathbb{R}^3\times S^2} B(\boldsymbol{n}\cdot(\boldsymbol{v}-\boldsymbol{v}_*), |\boldsymbol{v}-\boldsymbol{v}_*|)\,(\varphi'+\varphi'_*-\varphi-\varphi_*)ff_*d\mu dt\,.$$

for any test function $\varphi(x,\boldsymbol{v},t)$ which is twice differentiable as a function of $\boldsymbol{v}$ with second derivatives uniformly bounded with respect to x and t. In Eq.(3.1) we have used the notation

(3.2) $$d\mu = \sin\theta\, d\theta d\phi\, d\boldsymbol{v}_* d\boldsymbol{v} dx$$

We remark that for classical solutions the above definition is known to be equivalent to that in (2.2). The main reason for introducing it is that it may produce weak solutions (as opposed to renormalized solutions in the sense of DiPerna and Lions [8]) even if the collision term is not necessarily in L^1. It is surprising that the weak form has not been used before the recent work of the author [2, 4].

For a function f to be a weak solution of the Boltzmann equation, it must satisfy Eq.

(1.1), where the derivatives in the left hand side are distributional derivatives and the right hand side has been defined above.

Now we want to prove that the definition of the weak form of the collision term makes sense for inverse power potentials without introducing an angular cutoff, as first shown in a paper of the author [4]. To this end we consider the following identity:

$$\begin{aligned}(3.3)\qquad &\int_0^1 ds \int_0^1 dt \frac{\partial^2}{\partial s \partial t}[\varphi(\boldsymbol{v} + s(\boldsymbol{v}' - \boldsymbol{v}) + t(\boldsymbol{v}_* - \boldsymbol{v}'))] = \\ &\int_0^1 ds \left\{ \frac{\partial}{\partial s}[\varphi(\boldsymbol{v} + s(\boldsymbol{v}' - \boldsymbol{v}) + (\boldsymbol{v}_* - \boldsymbol{v}'))] - \frac{\partial}{\partial s}[\varphi(\boldsymbol{v} + s(\boldsymbol{v}' - \boldsymbol{v}))] \right\} = \\ &\varphi(\boldsymbol{v}_*) - \varphi(\boldsymbol{v}') - \varphi(\boldsymbol{v}'_*) + \varphi(\boldsymbol{v})\end{aligned}$$

Hence

$$(3.4)\qquad \varphi(\boldsymbol{v}) + \varphi(\boldsymbol{v}_*) - \varphi(\boldsymbol{v}') - \varphi(\boldsymbol{v}'_*) = \int_0^1 ds \int_0^1 dt \sum_{i,j=1}^{3} \frac{\partial^2 \varphi}{\partial v_i \partial v_j}(v'_i - v_i)(v^*_j - v'_j)$$

where the argument of φ is $\boldsymbol{v} + s(\boldsymbol{v}' - \boldsymbol{v}) + t(\boldsymbol{v}_* - \boldsymbol{v}')$. If K is an upper bound for the second derivatives, we obtain the following estimate

$$(3.5)\qquad |\varphi(\boldsymbol{v}) + \varphi(\boldsymbol{v}_*) - \varphi(\boldsymbol{v}') - \varphi(\boldsymbol{v}'_*)| \leq 9K|\boldsymbol{v}' - \boldsymbol{v}||\boldsymbol{v}^* - \boldsymbol{v}'| \leq |\boldsymbol{V}||\boldsymbol{n} \cdot \boldsymbol{V}|$$

Hence if the kernel B diverges for $\theta = \pi/2$, but $B \cos\theta$ is integrable, then the integral with respect to θ does not diverge. We recall that, if the intermolecular force varies as the n-th inverse power of the distance, then $B(\boldsymbol{n} \cdot (\boldsymbol{v} - \boldsymbol{v}_*), |\boldsymbol{v} - \boldsymbol{v}_*|)$ varies as shown in Refs. 2, 5.

We conclude that for power-law potentials, $B \cos\theta$ behaves as the power $-2/(n-1)$ of $|\pi/2 - \theta|$ and the definition of a weak solution given above makes sense for $n > 3$.

Henceforth we shall consider just Maxwell molecules, for which we state the main result of this section as

LEMMA 2.1: *The following estimate holds*

$$(3.6)\qquad \left| \int_{\mathbb{R}^3} Q(f,f)(x,\boldsymbol{v},t)\varphi(x,\boldsymbol{v},t)d\boldsymbol{v} \right| \leq \beta_0 K \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\boldsymbol{V}|^2 f f_* d\boldsymbol{v} d\boldsymbol{v}_* .$$

where K is un upper bound for the second derivatives of ϕ and β_0 a constant that only depends on molecular parameters.

4. - BASIC ESTIMATES

In this section and the next we shall be concerned with the initial value problem for the nonlinear Boltzmann equation when the solution depends on just one space coordinate which might range from $-\infty$ to $+\infty$ or from 0 to 1 (with periodicity boundary conditions); for definiteness we stick to the latter case. Easy modifications, in the vein of Ref. 6, are necessary to deal with the case of different boundary conditions. The x-, y- and z- component of the velocity $\boldsymbol{v} \in \mathbb{R}^3$ will be denoted by ξ, η and ζ respectively, and the Boltzmann equation reads

$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = Q(f,f) \tag{4.1}$$

with

$$Q(f,f)(x,\boldsymbol{v},t) = \int\!\!\int B(\boldsymbol{n}\cdot(\boldsymbol{v}-\boldsymbol{v}_*), |\boldsymbol{v}-\boldsymbol{v}_*|)\,(f'f'_* - ff_*)\,\sin\theta\, d\theta d\phi\, d\boldsymbol{v}_*. \tag{4.2}$$

We now set out to prove the crucial estimates for the solution of the initial value problem and for the collision term. It is safe to assume that we deal with a sufficiently regular solution of the problem, because this can always be enforced by truncating the collision kernel and modifying the collision terms in the way described in earlier work, in particular in Ref. 8. If we obtain strong enough bounds on the solutions of such truncated problems, we can then extract a subsequence converging to a renormalized solution in the sense of DiPerna and Lions; and the bounds which we do get actually guarantee that this solution is then a solution in the weak sense defined above.

Consider now the functional

$$I[f](t) = \int_{x<y} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\xi - \xi_*) f(x,\boldsymbol{v},t) f(y,\boldsymbol{v}_*,t)\, d\boldsymbol{v}_* d\boldsymbol{v}\, dx dy \tag{4.3}$$

where the integral with respect to x and y is over the triangle $0 \le x < y \le 1$. This functional was in the one-dimensional discrete velocity context first introduced by Bony[1]. The use of this functional is the main reason why we have to restrict our work to one dimension; no functional with similar pleasant properties is known, at this time, in more than one dimension (for a discussion of this point see a recent paper of the author [5]). Notice that if we have bounds for the integral with respect to x of $\rho = \int_{\mathbb{R}^3} f(x,\boldsymbol{v},t)\, d\boldsymbol{v}$ and for

$$E(t) = \int_0^1 \int |\boldsymbol{v}|^2 f \; d\boldsymbol{v}\, dx,$$

then we have control over the functional $I[f](t)$.

A short calculation with proper use of the collision invariants of the Boltzmann collision operator shows that

$$\frac{d}{dt} I[f] = - \int_{[0,1]} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\xi - \xi_*)^2 f(x,\boldsymbol{v}_*,t) f(x,\boldsymbol{v},t)\, d\boldsymbol{v} d\boldsymbol{v}_* dx \tag{4.4}$$

Notice that the term on the right, apart from the factor $(\xi - \xi_*)^2$, has structural similarity to the collision term of the Boltzmann equation, and the integrand is nonnegative. This is the reason why the functional $I[f]$ is a powerful tool.

After integration from 0 to $T > 0$ and reorganizing,

$$\int_0^T \int_{[0,1]} \int_{\boldsymbol{v}} \int_{\boldsymbol{v}_*} (\xi - \xi_*)^2 f(x, \boldsymbol{v}_*, t) f(x, \boldsymbol{v}, t)\, d\boldsymbol{v} d\boldsymbol{v}_* dx dt = I[f](0) - I[f](T). \tag{4.5}$$

According to a previous remark, the right-hand side of (4.5) is bounded. Since the total energy is conserved, we have proved

LEMMA 4.1: *If f is a sufficiently smooth solution of the initial value problem given by* (4.1) and (4.2) *with initial value f_0, then*

$$\int_0^t \int_0^1 \int_{\boldsymbol{v}} \int_{\boldsymbol{v}_*} (\xi - \xi_*)^2 f(x, \boldsymbol{v}_*, \tau) f(x, \boldsymbol{v}, \tau)\, d\boldsymbol{v} d\boldsymbol{v}_* dx d\tau$$

are bounded.

We have now the following

LEMMA 4.2: *Under the above assumptions, we have, for the weak solutions of the Boltzmann equation for noncutoff Maxwell molecules:*

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2 \times [0,T] \times [0,1]} |\boldsymbol{v} - \boldsymbol{v}_*|^2 f(x, \boldsymbol{v}, t) f(x, \boldsymbol{v}_*, t) B(\theta) dt d\mu < K_0 \tag{4.6}$$

where K_0 is a constant, which only depends on the initial data (and molecular constants).

In fact, we can take $\varphi = \xi^2$ as a test function and remark that the contribution of the left hand side is bounded in terms of the initial data because $\xi^2 \leq |\boldsymbol{v}|^2$. Hence the right hand side is also bounded. We can now replace $\boldsymbol{v}$ by $\boldsymbol{c} = \boldsymbol{v} - \boldsymbol{u}$ in the right hand side (since the extra terms vanish thanks to mass and momentum conservation). Then:

$$\varphi(\boldsymbol{v}) + \varphi(\boldsymbol{v}_*) - \varphi(\boldsymbol{v}') - \varphi(\boldsymbol{v}'_*) = 2n_1(\xi - \xi_*)\boldsymbol{n} \cdot (\boldsymbol{v} - \boldsymbol{v}_*) - 2|\boldsymbol{n} \cdot \boldsymbol{V}|^2. \tag{4.7}$$

When we integrate the first term with respect to $\boldsymbol{n}$ the contributions containing c_2 and c_3 vanish by symmetry and the weak collision term that remains to evaluate is

$$\int_{[0,T] \times [0,1] \times \mathbb{R}^3} Q(f,f)(x, \boldsymbol{v}, t) \xi^2 d\boldsymbol{v} dx dt = \tag{4.8}$$

$$- \int_{[0,T] \times [0,1] \times \mathbb{R}^3 \times \mathbb{R}^3 \times S^2} B(\theta) \{[n_1(\xi - \xi_*)]^2 - |\boldsymbol{n} \cdot \boldsymbol{V}|^2\}) f f_* d\mu dt\,.$$

We can separate the contributions from the two terms, since they separately converge and obtain

$$\int\limits_{[0,T]\times[0,1]\times\mathbb{R}^3} Q(f,f)(x,\boldsymbol{v},t)\xi^2 d\boldsymbol{v}dxdt = \tag{4.9}$$
$$- 3B_0 \int\limits_{[0,T]\times[0,1]\times\mathbb{R}^3\times\mathbb{R}^3} (\xi - \xi_*)^2 ff_* d\boldsymbol{v}d\boldsymbol{v}_* dxdt$$
$$+ B_0 \int\limits_{[0,T]\times[0,1]\times\mathbb{R}^3\times\mathbb{R}^3} |\boldsymbol{V}|^2 ff_* d\boldsymbol{v}d\boldsymbol{v}_* dxdt.$$

where if the force between two molecules at distance r is κr^{-5}, then

$$B_0 = a\sqrt{\frac{\kappa}{2m^3}} \quad (a = 1.3703\ldots). \tag{4.10}$$

The constant a was first computed by Maxwell [13]; the value given here was computed by Ikenberry and Truesdell [9]. Since we know that the left hand side of Eq. (4.9) is bounded and the first term in the right hand side is bounded, it follows that the last term is also bounded by a constant depending on initial data (and molecular constants, such as m and κ).

5. - Existence of weak solutions for noncutoff potentials

In order to prove the existence of a weak solution, we shall assume that this has been proved for Maxwell molecules with an angular cutoff [2]; actually to make the paper self-contained and the proof more explicit, we shall assume that the proof is available when a cutoff for small relative speeds is introduced. In this case, in fact the proof immediately follows from the DiPerna-Lions existence theorem with the estimate of Lemma 4.2; it is enough to remark that a solution exists when we renormalize by division by $1 + \varepsilon f$ (f independent of $\varepsilon > 0$, and we pass to the limit $\varepsilon \to 0$ thanks to (4.6).

In the noncutoff case we approximate the solution by cutting off the angles close to $\pi/2$ and the small relative speeds. In this way we can obtain a sequence f_n formally approximating the solution f whose existence we want to prove.

Lemma 5.1: *Let $\{f^n\}$ be a sequence of solutions to an approximating problem. There is a subsequence such that for each $T > 0$*

i) $\int f^n \, d\boldsymbol{v} \to \int f \, d\boldsymbol{v}$ a.e. and in $L^1((0,T)\times\mathbb{R}^3)$,

ii)
$$\int\limits_{\mathbb{R}^3} |\boldsymbol{V}|^2 f_{n*} d\boldsymbol{v}_* \to \int\limits_{\mathbb{R}^3} |\boldsymbol{V}|^2 f_* d\boldsymbol{v}_*$$

in $L^1((0,T)\times\mathbb{R}^3\times B_R)$ *for all* $R>0$*, and a.e.,*

iii)

$$(5.1)\qquad g_n(x,t)=\frac{\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}|\boldsymbol{V}|^2 f_n f_{n*}\,d\boldsymbol{v}\,d\boldsymbol{v}_*}{1+\int f_n\,d\boldsymbol{v}}\to\frac{\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}|\boldsymbol{V}|^2 f f_*\,d\boldsymbol{v}\,d\boldsymbol{v}_*}{1+\int f\,d\boldsymbol{v}}=g(x,t)$$

weakly in $L^1((0,T)\times(0,1))$.

Proof: i) is immediate. ii) uses an argument well-known in DiPerna-Lions proof with the estimate $\sup_n\int f_n(1+|\boldsymbol{v}|^2)\,d\boldsymbol{v}<\infty$ to reduce the problem to bounded domains with respect to $\boldsymbol{v}_*$.

For iii) we use i) and the fact that f_n converges weakly, but the factor multiplying it in the integral converges a.e. because of ii).

Now we remark that $g_n(x,t)$ converges weakly to $g(x,t)$ and $\rho_n(x,t)$ converges a.e. to $\rho(x,t)$ and the integral $\int\rho_n g_n dxdt$ is uniformly bounded to conclude with the following Lemma:

Lemma 5.2: *Let* $\{f_n\}$ *be a sequence of solutions to an approximating problem. There is a subsequence such that for each* $T>0$

$$(5.2)\qquad \int\limits_{(0,T)\times(0,1)\times\mathbb{R}^3\times\mathbb{R}^3}|\boldsymbol{V}|^2 f_n f_{n*}\,d\mu dt\to\int\limits_{(0,T)\times(0,1)\times\mathbb{R}^3\times\mathbb{R}^3}|\boldsymbol{V}|^2 f f_*\,d\mu dt$$

We can now prove the basic result:

Lemma 5.3: *Let* $\{f_n\}$ *be a sequence of solutions to an approximating problem, weakly converging to* f*. There is a subsequence such that for each* $T>0$

$$(5.3)\qquad \int\limits_{(0,T)\times(0,1)\times\mathbb{R}^3}\phi Q_n(f_n,f_n)\,dtdxd\boldsymbol{v}\to\int\limits_{(0,T)\times(0,1)\times\mathbb{R}^3}\phi Q(f,f)\,dtdxd\boldsymbol{v}$$

where Q_n *and* Q *are given by the weak form of the collision operator, as defined in Eq. (3.1).*

Proof: In fact the integrand in the left hand side of Eq. (5.3) is, thanks to (3.6), uniformly bounded by the integrand of Eq. (5.2) which weakly converges.

Thanks to this result, we can now pass to the limit in the approximating problem to obtain

Theorem 5.4: *Let* $f_0\in L^1(\mathbb{R}\times\mathbb{R}^3)$ *be such that*

$$(5.4)\qquad \int f_0(\,\cdot\,)(1+|\boldsymbol{v}|^2)\,d\boldsymbol{v}dx<\infty;\qquad \int f_0|\ln f_0(.)|\,d\boldsymbol{v}dx<\infty.$$

Then there is a weak solution $f(x, \boldsymbol{v}, t)$ *of the initial value problem* (1.1), (1.4), *such that* $f \in C(\mathbb{R}_+, L^1(\mathbb{R} \times \mathbb{R}^3)), f(.,0) = f_0$. *This solution conserves energy globally.*

6. - Concluding remarks

We have surveyed the existence theory of the nonlinear Boltzmann equation with particular attention for a recent result of the author concerning Maxwell molecules, without any truncation on the collision kernel, in the one-dimensional case. To the best of our knowledge, this is the first result for the noncutoff Boltzmann equation. The solution conserves energy globally.

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